

A first-order differential double subordination with applications[☆]

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Abstract

Let q_1 and q_2 belong to a certain class of normalized analytic univalent functions in the open unit disk of the complex plane. Sufficient conditions are obtained for normalized analytic functions p to satisfy the double subordination chain $q_1(z) \prec p(z) \prec q_2(z)$. The differential sandwich-type result obtained is applied to normalized univalent functions and to Φ -like functions.

Keywords: Differential subordination, differential superordination, best subordinant, best dominant, Φ -like functions

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1. Introduction

Let \mathcal{H} be the class consisting of analytic functions in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane \mathbb{C} . For $a \in \mathbb{C}$, let $\mathcal{H}[a, n] := \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\}$, and $\mathcal{A} := \{f \in \mathcal{H} : f(0) = 0, f'(0) = 1\}$. A function $f \in \mathcal{H}$ is said to be subordinate to an analytic function $g \in \mathcal{H}$, or g superordinates f , written $f(z) \prec g(z)$ ($z \in \mathbb{D}$), if there exists a Schwarz function w , analytic in \mathbb{D} with $w(0) = 0$ and $|w(z)| < 1$, satisfying $f(z) = g(w(z))$. If the function g is univalent in \mathbb{D} , then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. An exposition on the widely used theory of differential subordination, developed in the main by Miller and Mocanu, with numerous applications to univalent functions can be found in their monograph [14]. Miller and Mocanu [15] also introduced the dual concept of differential superordination. Let $p, h \in \mathcal{H}$ and $\phi(r, s, t; z) : \mathbb{C}^3 \times \mathbb{D} \rightarrow \mathbb{C}$. If p and $\phi(p(z), zp'(z), z^2 p''(z); z)$ are univalent and p satisfies the second-order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z), \quad (1)$$

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then p is a solution of the differential superordination (1). An analytic function q is called a *subordinant* if $q \prec p$ for all p satisfying (1). A univalent subordinant \tilde{q} satisfying $q \prec \tilde{q}$ for all subordinants q of (1) is said to be the best subordinant. Miller and Mocanu [15] obtained conditions on h , q and ϕ for which the following differential implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec p(z).$$

Using these results, Bulboacă gave a treatment on certain classes of first-order differential superordinations [6, 8], as well as superordination-preserving integral operators [7]. Ali *et al.* [2] gave several applications of first-order differential subordination and superordination to obtain sufficient conditions for normalized analytic functions f to satisfy $q_1(z) \prec zf'(z)/f(z) \prec q_2(z)$, where q_1 and q_2 are given univalent analytic functions in \mathbb{D} . In [3] they have also applied differential superordination to functions defined by means of linear operators. Recently Ali and Ravichandran [1] investigated first-order superordination to a class of meromorphic α -convex functions. Several differential subordination and superordination associated with various linear operators were also investigated in [4].

Generalizing the familiar starlike and convex functions, Lewandowski *et al.* [12] introduced γ -starlike functions consisting of $f \in \mathcal{A}$ satisfying the inequality

$$\operatorname{Re} \left(\left(\frac{zf'(z)}{f(z)} \right)^{1-\gamma} \left(1 + \frac{zf''(z)}{f'(z)} \right)^\gamma \right) > 0.$$

These functions are starlike. With $p(z) := zf'(z)/f(z)$, to show that γ -starlike functions are indeed starlike is to analytically make the implication

$$\operatorname{Re} \left(p(z) \left(1 + \frac{zp'(z)}{p^2(z)} \right)^\gamma \right) > 0 \Rightarrow \operatorname{Re} p(z) > 0.$$

Following the work of Lewandowski *et al.* [12, 13], Kanas *et al.* [10] determined conditions on p and h satisfying

$$p(z) \left(1 + \frac{zp'(z)}{p(z)} \right)^\alpha \prec h(z) \Rightarrow p(z) \prec h(z)$$

for a fixed $\alpha \in [0, 1]$. Lecko [11] (see Kanas *et al.* [10] for a symmetric version) investigated the more general subordination

$$p(z) \left(1 + \frac{zp'(z)}{p(z)} \varphi(p(z)) \right)^\alpha \prec h(z) \Rightarrow p(z) \prec h(z).$$

Singh and Gupta [17] subsequently investigated the following first-order differential subordination that included the important Briot-Bouquet differential subordination:

$$(p(z))^\alpha \left(p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \right)^\mu \prec (q(z))^\alpha \left(q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \right)^\mu \Rightarrow p(z) \prec q(z).$$

For a closely related class, see S. Kanas and J. Kowalczyk [9].

The present paper investigates differential subordination and superordination implications of expressions similar to the form considered above by Singh and Gupta [17]. Special cases of the results obtained include one involving the expression $\alpha p^2(z) + (1 - \alpha)p(z) + \alpha zp'(z)$, a result which cannot be deduced from the work of Singh and Gupta [17]. The sandwich-type results obtained in our present investigation are then applied to normalized analytic univalent functions and to Φ -like functions.

The following definition and results will be required:

Lemma 1.1 (cf. Miller and Mocanu [14, Theorem 3.4h, p.132]). *Let q be univalent in the unit disk \mathbb{D} , and let ϑ and ϕ be analytic in a domain $D \supset q(\mathbb{D})$ with $\phi(w) \neq 0$, $w \in q(\mathbb{D})$. With $Q(z) := zq'(z)\phi(q(z))$, let $h(z) := \vartheta(q(z)) + Q(z)$. Suppose that Q is starlike univalent in \mathbb{D} and*

$$\operatorname{Re} \left(\frac{zh'(z)}{Q(z)} \right) > 0 \quad (z \in \mathbb{D}).$$

If p is analytic in \mathbb{D} with $p(0) = q(0)$, $p(\mathbb{D}) \subset D$ and

$$\vartheta(p(z)) + zp'(z)\phi(p(z)) \prec \vartheta(q(z)) + zq'(z)\phi(q(z)),$$

then $p(z) \prec q(z)$, and q is the best dominant.

Definition 1.2 ([15, Definition 2, p. 817]). *Denote by \mathcal{Q} the set of all functions f that are analytic and injective on $\overline{\mathbb{D}} - E(f)$, where*

$$E(f) = \{ \zeta \in \partial\mathbb{D} : \lim_{z \rightarrow \zeta} f(z) = \infty \},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D} - E(f)$.

Lemma 1.3 ([8]). *Let q be univalent in the unit disk \mathbb{D} , ϑ and ϕ be analytic in a domain D containing $q(\mathbb{D})$. Suppose that $\operatorname{Re} [\vartheta'(q(z))/\phi(q(z))] > 0$ for $z \in \mathbb{D}$ and $zq'(z)\phi(q(z))$ is starlike univalent in \mathbb{D} . If $p \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$ with $p(\mathbb{D}) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\phi(p(z))$ is univalent in \mathbb{D} , then*

$$\vartheta(q(z)) + zq'(z)\phi(q(z)) \prec \vartheta(p(z)) + zp'(z)\phi(p(z))$$

implies $q(z) \prec p(z)$, and q is the best subdominant.

2. A Sandwich Theorem

Our main result involves the following class of functions:

Definition 2.1. *Let α and μ be fixed numbers with $0 < \mu \leq 1$, $\alpha + \mu \geq 0$. Also let β , γ and δ be complex numbers with $\beta \neq 0$. The class $\mathcal{R}(\alpha, \beta, \gamma, \delta, \mu)$ consists of analytic functions p with $p(0) = 1$, $p(z) \neq 0$ in \mathbb{D} , and are such that the functions*

$$P(z) := (p(z))^\alpha \left(p(z) + \delta + \frac{zp'(z)}{\beta p(z) + \gamma} \right)^\mu \quad (z \in \mathbb{D})$$

are well-defined in \mathbb{D} . (Here the powers are principal values.)

By making use of Lemma 1.1, the following result is derived:

Theorem 2.2. Let $q \in \mathcal{R}(\alpha, \beta, \gamma, \delta, \mu)$ be analytic and univalent in \mathbb{D} . Set

$$R(z) := \frac{zq'(z)}{\beta q(z) + \gamma} \quad (z \in \mathbb{D}). \quad (2)$$

Assume that

$$\operatorname{Re} \left((\beta q(z) + \gamma) \left(1 + \frac{\alpha}{\mu} + \frac{\alpha \delta}{\mu q(z)} \right) \right) > 0 \quad (z \in \mathbb{D}), \quad (3)$$

and

$$\operatorname{Re} \left(\frac{\alpha}{\mu} \frac{zq'(z)}{q(z)} + \frac{zR'(z)}{R(z)} \right) > 0 \quad (z \in \mathbb{D}). \quad (4)$$

If $p \in \mathcal{R}(\alpha, \beta, \gamma, \delta, \mu)$ satisfies

$$(p(z))^\alpha \left(p(z) + \delta + \frac{zp'(z)}{\beta p(z) + \gamma} \right)^\mu \prec (q(z))^\alpha \left(q(z) + \delta + \frac{zq'(z)}{\beta q(z) + \gamma} \right)^\mu, \quad (5)$$

then $p(z) \prec q(z)$, and q is the best dominant.

Proof. We first write the differential subordination (5) as

$$(p(z))^{\frac{\alpha}{\mu}+1} + \delta(p(z))^{\frac{\alpha}{\mu}} + (p(z))^{\frac{\alpha}{\mu}} \frac{zp'(z)}{\beta p(z) + \gamma} \prec (q(z))^{\frac{\alpha}{\mu}+1} + \delta(q(z))^{\frac{\alpha}{\mu}} + (q(z))^{\frac{\alpha}{\mu}} \frac{zq'(z)}{\beta q(z) + \gamma}.$$

Define the functions ϑ and φ by

$$\vartheta(w) := w^{\frac{\alpha}{\mu}+1} + \delta w^{\frac{\alpha}{\mu}} \text{ and } \varphi(w) := \frac{w^{\frac{\alpha}{\mu}}}{\beta w + \gamma}.$$

Since $q \in \mathcal{R}(\alpha, \beta, \gamma, \delta, \mu)$, then $q(z) \neq 0$ and therefore $\varphi(w) \neq 0$ when $w \in q(\mathbb{D})$. Also φ and ϑ are analytic in a domain containing $q(\mathbb{D})$. Define the function

$$Q(z) := zq'(z)\varphi(q(z)) = (q(z))^{\frac{\alpha}{\mu}} \frac{zq'(z)}{\beta q(z) + \gamma} = (q(z))^{\frac{\alpha}{\mu}} R(z),$$

where R is given by (2). It follows from (4) that

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} = \Re \left(\frac{\alpha}{\mu} \frac{zq'(z)}{q(z)} + \frac{zR'(z)}{R(z)} \right) > 0,$$

and so Q is a starlike function. Now define h by

$$h(z) := \vartheta(q(z)) + Q(z) = (q(z))^{\frac{\alpha}{\mu}+1} + \delta(q(z))^{\frac{\alpha}{\mu}} + Q(z).$$

In view of the assumptions (3) and (4), it follows that

$$\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re} \left\{ (\beta q(z) + \gamma) \left(1 + \frac{\alpha}{\mu} + \frac{\alpha \delta}{\mu q(z)} \right) + \frac{\alpha}{\mu} \frac{zq'(z)}{q(z)} + \frac{zR'(z)}{R(z)} \right\} > 0 \quad (z \in \mathbb{D}).$$

The result is now deduced from Lemma 1.1. □

Example 2.3. Let $q : \mathbb{D} \rightarrow \mathbb{C}$ be defined by $q(z) = (1 + Az)/(1 + Bz)$ with $-1 < B < A \leq 1$. It is evident that $q \in \mathcal{R}(\alpha, \beta, \gamma, \delta, \mu)$ whenever

$$\delta + \frac{1-A}{1-B} > \frac{A-B}{(1-B)|\beta + \gamma| - |\beta A + \gamma B|}.$$

With additional constraints on the parameters, there exists functions q satisfying the hypothesis of Theorem 2.2. For instance, in addition to the above condition, assuming all the parameters $\alpha, \beta, \gamma, \delta$, and μ are positive with

$$\frac{1-2A}{1-A} > \frac{|\beta A + \gamma B|}{|\beta + \gamma| - |\beta A + \gamma B|},$$

then q satisfies the conditions of Theorem 2.2.

By a similar application of Lemma 1.3, the following result can be established, which we state without proof.

Theorem 2.4. Let $q \in \mathcal{R}(\alpha, \beta, \gamma, \delta, \mu)$ be as in Theorem 2.2. Let $p \in \mathcal{R}(\alpha, \beta, \gamma, \delta, \mu)$ satisfies $p \in \mathcal{H} \cap \mathcal{Q}$ and $(p(z))^{\frac{\alpha}{\mu}+1} + \delta(p(z))^{\frac{\alpha}{\mu}} + (p(z))^{\frac{\alpha}{\mu}} \frac{zp'(z)}{\beta p(z) + \gamma}$ be univalent. If p satisfies

$$(q(z))^\alpha \left(q(z) + \delta + \frac{zq'(z)}{\beta q(z) + \gamma} \right)^\mu \prec (p(z))^\alpha \left(p(z) + \delta + \frac{zp'(z)}{\beta p(z) + \gamma} \right)^\mu,$$

then $q(z) \prec p(z)$, and q is the best subdominant.

Combining Theorems 2.2 and 2.4, the following “sandwich theorem” is obtained:

Theorem 2.5. Let $q_i \in \mathcal{R}(\alpha, \beta, \gamma, \delta, \mu)$ ($i = 1, 2$) be analytic and univalent in \mathbb{D} . Set

$$R_i(z) := \frac{zq'_i(z)}{\beta q_i(z) + \gamma} \quad (i = 1, 2; z \in \mathbb{D}),$$

$$h_i(z) := (q_i(z))^\alpha \left(q_i(z) + \delta + \frac{zq'_i(z)}{\beta q_i(z) + \gamma} \right)^\mu \quad (i = 1, 2).$$

Assume that

$$\operatorname{Re} \left((\beta q_i(z) + \gamma) \left(1 + \frac{\alpha}{\mu} + \frac{\alpha \delta}{\mu q_i(z)} \right) \right) > 0 \quad (z \in \mathbb{D})$$

and

$$\operatorname{Re} \left(\frac{\alpha}{\mu} \frac{zq'_i(z)}{q_i(z)} + \frac{zR'_i(z)}{R_i(z)} \right) > 0 \quad (i = 1, 2; z \in \mathbb{D}).$$

If $p \in \mathcal{R}(\alpha, \beta, \gamma, \delta, \mu)$ satisfies $p \in \mathcal{H} \cap \mathcal{Q}$ and $(p(z))^{\frac{\alpha}{\mu}+1} + \delta(p(z))^{\frac{\alpha}{\mu}} + (p(z))^{\frac{\alpha}{\mu}} \frac{zp'(z)}{\beta p(z) + \gamma}$ is univalent, then

$$h_1(z) \prec (p(z))^\alpha \left(p(z) + \delta + \frac{zp'(z)}{\beta p(z) + \gamma} \right)^\mu \prec h_2(z) \quad (6)$$

implies $q_1(z) \prec p(z) \prec q_2(z)$. Further q_1 and q_2 are respectively the best subdominant and the best dominant.

3. Applications to Univalent Functions

By use of Theorem 2.5, the following result is obtained:

Theorem 3.1. Let α, μ be fixed numbers with $0 < \mu \leq 1$, $\alpha + \mu > 0$, and $\lambda \in \mathbb{C}$. Let $f, g \in \mathcal{A}$, and $q_i(z) = zg'_i(z)/g_i(z)$ ($i = 1, 2$) be univalent in \mathbb{D} satisfying

$$\operatorname{Re} \left(\frac{1}{\lambda} q_i(z) \right) > 0$$

and

$$\operatorname{Re} \left(\left(\frac{\alpha}{\mu} - 1 \right) \frac{zq'_i(z)}{q_i(z)} + 1 + \frac{zq''_i(z)}{q'_i(z)} \right) > 0.$$

Let

$$h_i(z) := \left(\frac{zg'_i(z)}{g_i(z)} \right)^\alpha \left((1 - \lambda) \frac{zg'_i(z)}{g_i(z)} + \lambda \left(1 + \frac{zg''_i(z)}{g'_i(z)} \right) \right)^\mu \quad (i = 1, 2).$$

If $f \in \mathcal{A}$ satisfies $0 \neq \frac{zf'(z)}{f(z)} \in \mathcal{H}[1, 1] \cap \mathcal{Q}$ and $\left(\frac{zf'(z)}{f(z)} \right)^\alpha \left((1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left(1 + \frac{zf''(z)}{f'(z)} \right) \right)^\mu$ is univalent in \mathbb{D} , then

$$h_1(z) \prec \left(\frac{zf'(z)}{f(z)} \right)^\alpha \left((1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left(1 + \frac{zf''(z)}{f'(z)} \right) \right)^\mu \prec h_2(z)$$

implies

$$\frac{zg'_1(z)}{g_1(z)} \prec \frac{zf'(z)}{f(z)} \prec \frac{zg'_2(z)}{g_2(z)}.$$

Proof. The result follows from Theorem 2.5 by taking $\gamma = \delta = 0$, $\beta = 1/\lambda$, and

$$p(z) := \frac{zf'(z)}{f(z)} \text{ and } q_i(z) := \frac{zg'_i(z)}{g_i(z)} \quad (i = 1, 2). \quad \square$$

The following two corollaries are immediate consequences of Theorem 2.5 (or Theorem 3.1):

Corollary 3.2. [2] Let $\alpha \in \mathbb{C}$, and $q_i(z) \neq 0$ ($i = 1, 2$) be univalent in \mathbb{D} . Assume that $\operatorname{Re} [\overline{\alpha} q_i(z)] > 0$ for $i = 1, 2$ and $zq'_i(z)/q_i(z)$ ($i = 1, 2$) is starlike univalent in \mathbb{D} . If $f \in \mathcal{A}$, $0 \neq zf'(z)/f(z) \in \mathcal{H}[1, 1] \cap \mathcal{Q}$, $(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right)$ is univalent in \mathbb{D} , then

$$q_1(z) + \alpha \frac{zq'_1(z)}{q_1(z)} \prec (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) \prec q_2(z) + \alpha \frac{zq'_2(z)}{q_2(z)}$$

implies

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z).$$

Further q_1 and q_2 are respectively the best subdominant and best dominant.

Corollary 3.3. [2] Let $q_i(z) \neq 0$ be univalent in \mathbb{D} with $\operatorname{Re} q_i(z) > 0$. Let $zq'_i(z)/q_i^2(z)$ be starlike univalent in \mathbb{D} for $i = 1, 2$. If $f \in \mathcal{A}$, $0 \neq zf'(z)/f(z) \in \mathcal{H}[1, 1] \cap \mathcal{Q}$, $\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)}$ is univalent in \mathbb{D} , then

$$1 + \frac{zq'_1(z)}{q_1^2(z)} \prec \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 + \frac{zq'_2(z)}{q_2^2(z)}$$

implies $q_1(z) \prec zf'(z)/f(z) \prec q_2(z)$. Further q_1 and q_2 are respectively the best subordinant and best dominant.

Another application of Theorem 2.5 yields the following result:

Corollary 3.4. [2] Let q_1 and q_2 be convex univalent in \mathbb{D} . Let $0 \neq \alpha \in \mathbb{C}$, and assume that $\operatorname{Re} q_i(z) > \operatorname{Re} \frac{\alpha-1}{2\alpha}$ for $i = 1, 2$. If $f \in \mathcal{A}$, $zf'(z)/f(z) \in \mathcal{H}[1, 1] \cap \mathcal{Q}$, $\frac{zf'(z)}{f(z)} + \alpha \frac{zf''(z)}{f'(z)}$ is univalent in \mathbb{D} , then

$$(1 - \alpha)q_1(z) + \alpha q_1^2(z) + \alpha zq'_1(z) \prec \frac{zf'(z)}{f(z)} \left(1 + \alpha \frac{zf''(z)}{f'(z)} \right) \prec (1 - \alpha)q_2(z) + \alpha q_2^2(z) + \alpha zq'_2(z)$$

implies

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z).$$

Further q_1 and q_2 are respectively the best subordinant and best dominant.

4. Application to Φ -like Functions

Let Φ be an analytic function in a domain containing $f(\mathbb{D})$, $\Phi(0) = 0$ and $\Phi'(0) > 0$. A function $f \in \mathcal{A}$ is called Φ -like if

$$\operatorname{Re} \frac{zf'(z)}{\Phi(f(z))} > 0 \quad (z \in \mathbb{D}).$$

This concept was introduced by Brickman [5] and it was shown that an analytic function $f \in \mathcal{A}$ is univalent if and only if f is Φ -like for some Φ . When $\Phi(w) = w$ and $\Phi(w) = \lambda w$, the Φ -like function f is respectively starlike and spirallike of type $\arg \lambda$. Ruscheweyh [16] introduced and studied the following general class of Φ -like functions:

Definition 4.1. Let Φ be analytic in a domain containing $f(\mathbb{D})$, $\Phi(0) = 0$, $\Phi'(0) = 1$ and $\Phi(\omega) \neq 0$ for $\omega \in f(\mathbb{D}) - \{0\}$. Let q be a fixed analytic function in \mathbb{D} , with $q(0) = 1$. A function $f \in \mathcal{A}$ is called Φ -like with respect to q if

$$\frac{zf'(z)}{\Phi(f(z))} \prec q(z) \quad (z \in \mathbb{D}).$$

Theorem 4.2. Let $\alpha \neq 0$ be a complex number and q_i ($i = 1, 2$) be convex univalent in \mathbb{D} . Define h_i by

$$h_i(z) := \alpha q_i^2(z) + (1 - \alpha)q_i(z) + \alpha z q_i'(z) \quad (i = 1, 2),$$

and suppose that

$$\operatorname{Re} \left(\frac{1 - \alpha}{\alpha} + 2q_i(z) \right) > 0 \quad (i = 1, 2; z \in \mathbb{D}).$$

If $f \in \mathcal{A}$ satisfies $f \in \mathcal{H}[1, 1] \cap \mathcal{Q}$ and $\frac{zf'(z)}{\Phi(f(z))} \left(1 + \frac{\alpha z f''(z)}{f'(z)} + \frac{\alpha z(f'(z) - (\Phi(f(z)))')}{\Phi(f(z))} \right)$ is univalent in \mathbb{D} , then

$$h_1(z) \prec \frac{zf'(z)}{\Phi(f(z))} \left(1 + \frac{\alpha z f''(z)}{f'(z)} + \frac{\alpha z(f'(z) - (\Phi(f(z)))')}{\Phi(f(z))} \right) \prec h_2(z) \quad (7)$$

implies

$$q_1(z) \prec \frac{zf'(z)}{\Phi(f(z))} \prec q_2(z).$$

Further q_1 and q_2 are respectively the best subordinant and the best dominant.

Proof. Define the function p by

$$p(z) := \frac{zf'(z)}{\Phi(f(z))} \quad (z \in \mathbb{D}). \quad (8)$$

Then the function p is analytic in \mathbb{D} with $p(0) = 1$. From (8), it follows that

$$\begin{aligned} & \frac{zf'(z)}{\Phi(f(z))} \left(1 + \frac{\alpha z f''(z)}{f'(z)} + \frac{\alpha z(f'(z) - (\Phi(f(z)))')}{\Phi(f(z))} \right) \\ &= p(z) \left(1 + \alpha \left(\frac{zp'(z)}{p(z)} - 1 \right) + \alpha p(z) \right) \\ &= \alpha p^2(z) + (1 - \alpha)p(z) + \alpha zp'(z). \end{aligned} \quad (9)$$

Putting (9) in the subordination (7) yields

$$h_1(z) \prec \alpha p^2(z) + (1 - \alpha)p(z) + \alpha zp'(z) \prec h_2(z).$$

The result now follows from Theorem 2.5. □

Remark 1. When $\Phi(w) = w$, Theorem 4.2 reduces to Corollary 3.4.

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